14.3. Partial derivatives

Recall: Given a function $f(x)$, its derivative at $x=a$ is defined by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

"change rate of f"

Def Given a function $f(x,y)$, its partial derivatives at (a,b) are defined by

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) := \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

"change rate in the x-direction"

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) := \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

"change rate in the y-direction".

Note Given a function $f(x,y,z)$ of three variables, you can similarly define its partial derivatives $f_x(a,b,c)$, $f_y(a,b,c)$, $f_z(a,b,c)$

Recall : Given a function $f(x)$, we derive it multiple times to get higher derivatives.

e.g. $f'' = (f')'$, $f''' = (f'')'$, ...

Note Given a function $f(x,y)$, we get higher partial derivatives.

e.g. $f_{xx} = (f_x)_x = \dfrac{\partial f_x}{\partial x}$, $f_{xy} = (f_x)_y = \dfrac{\partial f_x}{\partial y}$,

$f_{xxy} = (f_{xx})_y = \dfrac{\partial f_{xx}}{\partial y}$, ...

Thm If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Note In Math 215, essentially all multi-variable functions will have this property. Hence the order of partial differentiation does not matter for us.

e.g. $f_{xy} = f_{yx}$, $f_{xxy} = f_{xyx} = f_{yxx}$, ...

Ex Some values of $f(x,y)$ are given as follows:

$x \backslash y$	2	2.5	3	3.5	4
1.8	11	15	18	20	21
2	10	13	16	19	20
2.2	7	9	14	16	17

(1) Estimate $f_x(2,3)$ and $f_y(2,3)$

Sol $f_x(2,3) \approx$ change rate from $(1.8, 3)$ to $(2.2, 3)$

$$= \frac{f(2.2, 3) - f(1.8, 3)}{2.2 - 1.8}$$

$$= \frac{14 - 18}{0.4} = \boxed{-10}$$

$f_y(2,3) \approx$ change rate from $(2, 2.5)$ to $(2, 3.5)$

$$= \frac{f(2, 3.5) - f(2, 2.5)}{3.5 - 2.5}$$

$$= \frac{19 - 13}{1} = \boxed{6.}$$

(2) Estimate $f_{xy}(2,3)$

<u>Sol</u> $f_{xy}(2,3) \approx$ change rate of f_x from $(2,2.5)$ to $(2,3.5)$

$$= \frac{f_x(2,3.5) - f_x(2,2.5)}{3.5 - 2.5}$$

$f_x(2,3.5) \approx$ change rate of f from $(1.8,3.5)$ to $(2.2,3.5)$

$$= \frac{f(2.2,3.5) - f(1.8,3.5)}{2.2 - 1.8}$$

$$= \frac{16 - 20}{0.4} = -10$$

$f_x(2,2.5) \approx$ change rate of f from $(1.8,2.5)$ to $(2.2,2.5)$

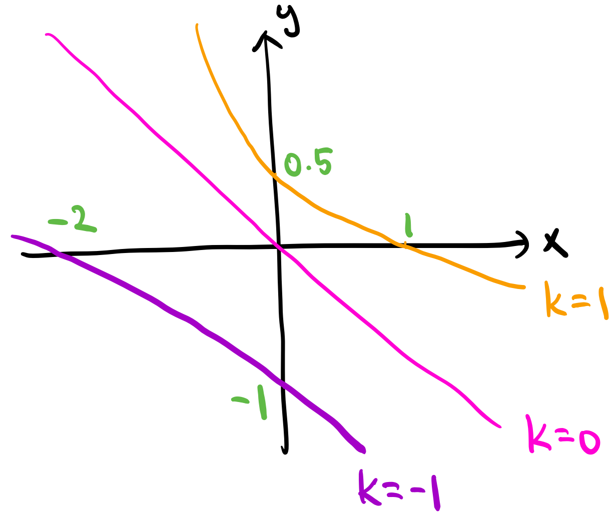
$$= \frac{f(2.2,2.5) - f(1.8,2.5)}{2.2 - 1.8}$$

$$= \frac{9 - 15}{0.4} = -15$$

$$\Rightarrow f_{xy}(2,3) = \frac{-10 - (-15)}{1} = \boxed{5}$$

<u>Note</u> You can instead estimate $f_{yx}(2,3)$

Ex A contour map of $g(x,y)$ is given as follows:



(1) Estimate $g_x(0,0)$.

Sol $g_x(0,0) \approx$ change rate from $(-2,0)$ to $(1,0)$

$$= \frac{g(1,0) - g(-2,0)}{1-(-2)}$$

$$= \frac{1-(-1)}{3} = \boxed{\frac{2}{3}}$$

Note You will often have to estimate the coordinates of points on a level curve.

(2) Find the sign of $g_y(0,0)$

Sol At $(0,0)$, the function g is increasing in the y-direction (moving to higher levels)

$\Rightarrow g_y(0,0)$ is $\boxed{\text{positive}}$

<u>Ex</u> Find all second order partial derivatives of

$$h(x,y) = \ln(x+y^2)$$

<u>Sol</u> When you take the partial derivative with respect to one variable, you regard all other variables as constants.

$$h_x = \underbrace{\frac{\partial}{\partial x}}_{y\text{ constant}}(\ln(x+y^2)) \underset{\text{chain rule}}{=} \frac{1}{x+y^2}\cdot\frac{\partial}{\partial x}(x+\cancelto{0}{y^2}) = \frac{1}{x+y^2}.$$

$$h_y = \underbrace{\frac{\partial}{\partial y}}_{x\text{ constant}}(\ln(x+y^2)) \overset{\text{chain rule}}{=} \frac{1}{x+y^2}\cdot\frac{\partial}{\partial y}(\cancelto{0}{x}+y^2) = \frac{2y}{x+y^2}.$$

$$h_{xx} = \frac{\partial h_x}{\partial x} = \underbrace{\frac{\partial}{\partial x}}_{y\text{ constant}}\left(\frac{1}{x+y^2}\right) = \frac{\partial}{\partial x}\left((x+y^2)^{-1}\right)$$

$$\underset{\text{chain rule}}{=} -(x+y^2)^{-2}\cdot\frac{\partial}{\partial x}(x+\cancelto{0}{y^2}) = \boxed{-\frac{1}{(x+y^2)^2}}$$

$$h_{xy} = \frac{\partial h_x}{\partial y} = \underbrace{\frac{\partial}{\partial y}}_{x\text{ constant}}\left(\frac{1}{x+y^2}\right) = \frac{\partial}{\partial y}\left((x+y^2)^{-1}\right)$$

$$\underset{\text{chain rule}}{=} -(x+y^2)^{-2}\cdot\frac{\partial}{\partial y}(\cancelto{0}{x}+y^2) = \boxed{-\frac{2y}{(x+y^2)^2}}$$

$$h_{yx} = h_{xy} = \boxed{-\frac{2y}{(x+y^2)^2}}$$

* You can also directly compute this

$$h_{yy} = \frac{\partial h_y}{\partial y} = \underline{\frac{\partial}{\partial y}}\left(\frac{2y}{x+y^2}\right)$$

x constant

$$= \frac{\frac{\partial}{\partial y}(2y)\cdot(x+y^2) - 2y\cdot\frac{\partial}{\partial y}(\cancel{x}^{0}+y^2)}{(x+y^2)^2}$$

quotient rule

$$= \frac{2(x+y^2) - 2y\cdot 2y}{(x+y^2)^2} = \boxed{\frac{2x-2y^2}{(x+y^2)^2}}$$